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Synchronization of Uncertain Euler-Lagrange Systems with Unknown Time-Varying Communication Delays

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***Abstract*— A decentralized controller is presented along with sufficient conditions for stability in leader-based synchronization of communication-delayed networked agents. The agents have heterogeneous dynamics modeled by uncertain, nonlinear Euler-Lagrange equations of motion affected by heterogeneous, unknown, exogenous disturbances. The developed controller requires only one-hop (delayed) communication from network neighbors and the communication delays are assumed to be heterogeneous, uncertain and time-varying. The presented approach uses a Lyapunov-based stability analysis in conjunction with Lyapunov-Krasovskii functionals to provide sufficient conditions which depend on the upper bound of the heterogeneous delays, feedback gains, and network connectivity, among other factors.**

I. Introduction

Synchronization is a type of cooperative control for networked systems in which autonomous agents act independently to accomplish a network-wide goal, and generally refers to matching the states of networked dynamical systems (cf. [1]–[4]). Example synchronization applications include power networks, collective satellite interferometry, and surveillance by autonomous vehicles (cf. [3], [5]). Synchronizing controllers are typically developed based on a decentralized interaction framework, in which agents use sensing or communication with network neighbors to compute a control policy. Network leaders can be included in synchronization applications so that the networked "follower" agents synchronize to some useful state trajectory instead of a constant consensus value dependent on initial conditions, where the network leader agent may simply be a preset time-varying trajectory or state of a physical system with which the follower agents interact. Restricting interaction with the leader agent to a strict subset of the follower agents provides a framework which is more applicable in practical scenarios.

Communication delay, also known as broadcast, coupling or transmission delay, is a phenomenon in which inter-agent interaction is delayed during information exchange. Even a small communication delay, such as that caused by information processing or a communication protocol, can cause networked autonomous systems to become unstable (cf. [5]), and hence, analysis is motivated to ensure stability of the control objective. Controllers developed in [6]–[15] are designed to provide stability for a network of communication-delayed autonomous synchronizing agents without the presence of a network leader. As demonstrated in [6], asymptotic convergence towards a fixed consensus point is achievable, despite the effects of the communication delay, for synchronization without a leader. The communication-delayed synchronization problem is generalized in [3], [16]–[18] to include a network leader, wherein every follower agent interacts with the leader agent. As illustrated in [3], asymptotic convergence towards the leader trajectory is achievable for synchronizing agents with full leader connectivity, despite the effects of communication delay.

Synchronization with a time-varying leader trajectory and limited leader connectivity presents a more challenging problem: if an agent is not aware of the leader's state, it must depend on the delayed state of neighboring follower agent(s), i.e., the effect of a change in the leader's state may not affect a follower agent until the time duration of multiple communication delays has passed. The controllers in [19]–[21] are developed to address this more challenging method of communication-delayed synchronization. The work in [19] is developed for follower agents with single integrator dynamics, undelayed state communication and uniformly delayed communication of control effort. The controller in [20] is designed for follower agents with single integrator dynamics and uniformly delayed state communication. Synchronization with uniformly delayed state communication is investigated in [21] for follower agents with nonlinear dynamics; however, the development assumes that the follower agents' dynamics are globally Lipschitz, which is restrictive and excludes many physical and electrical systems. Because globally Lipschitz dynamics can be uniformly upper-bounded by a linear expression, the result in [21] develops a stability analysis which does not account for general nonlinearities. Hence, the developments in [19]–[21] do not directly apply to networks with agents which have general nonlinear dynamics. A new strategy is required for demonstrating stability in synchronization of a network of agents with general nonlinear dynamics, delayed communication, and limited connectivity to a time-varying leader trajectory.

This paper considers the problem of synchronization of

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a leader-follower network of agents with heterogeneous dynamics described by nonlinear Euler-Lagrange equations of motion affected by an unknown, time-varying, exogenous input disturbance. The leader agent has a time-varying trajectory and is assumed to interact with at least one follower agent. The follower agents are delayed in communicating state information and do not communicate control effort information. The communication delay is assumed to be uncertain, heterogeneous, time-varying and bounded. A Lyapunov-based stability analysis using Lyapunov-Krasovskii (LK) functionals is provided to develop sufficient conditions for uniformly ultimately bounded (UUB) convergence to the leader state for each follower agent.

II. Problem Formulation

A. Graph theory preliminaries

Consider a network with one leader and a finite number $F \in \mathbb{Z}_{>0}$ of follower agents. The interaction among the follower agents is described by a fixed undirected graph $\mathcal{G}_F = \{\mathcal{V}_F, \mathcal{E}_F\}$, where $\mathcal{V}_F \triangleq \{1, \ldots, F\}$ is the node set representing the follower agents and $\mathcal{E}_F \subseteq \mathcal{V}_F \times \mathcal{V}_F$ is an edge set representing the communication links among the follower agents. An undirected edge, represented by the pair (j, i), belongs to $\mathcal{E}_F$ if agents $i, j \in \mathcal{V}_F$ communicate with each other. The neighbor set in $\mathcal{G}_F$ for agent $i \in \mathcal{V}_F$ is defined as $\mathcal{N}_{Fi} \triangleq \{j \in \mathcal{V}_F \mid (j, i) \in \mathcal{E}_F\}$. Connections in $\mathcal{G}_F$ are described by the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{F \times F}$, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}_F$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L}_F \in \mathbb{R}^{F \times F}$ of graph $\mathcal{G}_F$ is defined as $\mathcal{L}_F \triangleq D - A$, where $D \triangleq \operatorname{diag}\left\{\sum_{j \in \mathcal{N}_{F1}} a_{1j}, \ldots, \sum_{j \in \mathcal{N}_{FF}} a_{Fj}\right\}$ is the degree matrix. A leader-included directed supergraph of $\mathcal{G}_F$ can be constructed as $\mathcal{G} = \{\mathcal{V}_F \cup L, \mathcal{E}_F \cup \mathcal{E}_L\}$, where the node L represents the leader agent and the ordered pair $(i, L) \in \mathcal{E}_L$ if and only if agent $i \in \mathcal{V}_F$ receives information from the leader. The diagonal leader-connectivity matrix $B = \operatorname{diag}\{b_1, \ldots, b_F\} \in \mathbb{R}^{F \times F}$ is defined such that $b_i > 0$ if $(i, L) \in \mathcal{E}_L$ and $b_i = 0$ otherwise.

Assumption 1. The graph $\mathcal{G}_F$ is connected and at least one follower agent $i \in \mathcal{V}_F$ receives information from the leader.

Throughout the rest of this paper, $\mathcal{L}_F$ and B will be used to succinctly describe the interactions among the follower agents and the interactions between the follower agents and the leader. For brevity, let $L_B \triangleq (\mathcal{L}_F + B) \otimes I_m \in \mathbb{R}^{Fm \times Fm}$, where m is the dimension of the subsequently introduced state and $\otimes$ denotes the Kronecker product. Provided Assumption 1 is satisfied, then L_B is symmetric and positive definite [22].

B. Dynamic Model and Properties

Let the dynamics of follower agent $i \in \mathcal{V}_F$ be represented by Euler-Lagrange equations of motion of the form

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + F_i(\dot{q}_i) + G_i(q_i) = u_i + d_i(t), \quad (1)$$

where $q_i \in \mathbb{R}^m$ is the generalized configuration coordinate, $M_i : \mathbb{R}^m \to \mathbb{R}^{m \times m}$ is the inertia matrix, $C_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$ is the Coriolis/centrifugal matrix, $F_i : \mathbb{R}^m \to \mathbb{R}^m$ represents the effects of friction, $G_i : \mathbb{R}^m \to \mathbb{R}^m$ represents gravitational torques, $u_i \in \mathbb{R}^m$ is the vector of control inputs, and $d_i : \mathbb{R} \to \mathbb{R}^m$ is the time-varying, unknown, exogenous input disturbance. The time-varying state of the leader is denoted by $q_L : \mathbb{R} \to \mathbb{R}^m$, which is communicated to at least one of the follower agents. For simplicity in analysis, the following property and assumptions are used concerning the Euler-Lagrange dynamics, external disturbance and leader trajectory.

Property 1. [23] For each follower agent $i \in \mathcal{V}_F$, the inertia matrix is positive definite and symmetric, and there exist positive constants $\underline{m}, \bar{m} \in \mathbb{R}$ such that the inertia matrix satisfies the inequality $\underline{m}\|\xi\|^2 \leq \xi^T M_i(\psi)\xi \leq \bar{m}\|\xi\|^2$ for all $\psi, \xi \in \mathbb{R}^m$ and $i \in \mathcal{V}_F$.

Assumption 2. [24] For each follower agent $i \in \mathcal{V}_F$, the dynamics are sufficiently smooth such that the functions M_i, C_i, F_i, G_i are first-order differentiable, i.e., the first-order derivative is bounded if $q_i, \dot{q}_i, \ddot{q}_i \in \mathcal{L}_\infty$.

Assumption 3. For each follower agent $i \in \mathcal{V}_F$, the vector of time-varying input disturbances is continuous and bounded such that $\sup_{t \in \mathbb{R}} \|d_i(t)\| \leq \bar{d}$ for some positive constant $\bar{d} \in \mathbb{R}$.

Assumption 4. The leader trajectory q_L is sufficiently smooth such that $q_L \in \mathcal{C}^2$ and bounded such that $q_L, \dot{q}_L, \ddot{q}_L \in \mathcal{L}_\infty$.

The communication delay between follower agents is represented such that if the controller u_i is a function of a neighbor's state $q_j \in \mathcal{V}_F$, then the controller at time t only has access to the information $q_j(s)$, $s \leq t - \tau_{ji}(t)$ where $\tau_{ji}(t) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is the positive, time-varying, uncertain communication delay; i.e., at time t, agent $i \in \mathcal{V}_F$ is not aware of the state of neighbor $j \in \mathcal{V}_F$ $(i \neq j)$ from the time $t - \tau_{ji}(t)$ to t. The communication delays in the network need not be homogenous, i.e., the communication delays may be different for each interaction link. The communication delay may even differ between an interacting pair of agents, i.e., it may be that $\tau_{ij}(t) \neq \tau_{ji}(t)$ for $i, j \in \mathcal{V}_F$. The following assumption specifies the class of delays considered in this paper.

Assumption 5. The unknown, time-varying delay τ_{ji} is bounded above by a known constant $\bar{\tau} \in \mathbb{R}_{>0}$ such that $\sup_{t \in \mathbb{R}} \tau_{ji}(t) < \bar{\tau}$, τ_{ji} is differentiable, and τ_{ji} changes sufficiently slowly such that $\sup_{t \in \mathbb{R}} |\dot{\tau}_{ji}(t)| < 1$, for every $(j, i) \in \mathcal{E}_F$. Additionally, there is no delay in access to the leader's state, q_L, and no delay in agent $i \in \mathcal{V}_F$ knowing its own state, q_i.

For implementation purposes, it is also assumed that for every agent $i \in \mathcal{V}_F$, the delayed state $q_j(t - \tau_{ji}(t))$ has been communicated to agent i from every neighbor $j \in \mathcal{N}_{Fi}$ for at least $\bar{\tau}$ seconds before control implementation.

C. Control objective

The network-wide objective is to cooperatively drive the states of the networked agents towards the state of the network leader such that $\|q_L(t) - q_i(t)\| \to 0$ as $t \to \infty$ for all $i \in \mathcal{V}_F$ using decentralized one-hop communication, despite modeling uncertainties; exogenous disturbances; unknown, heterogeneous, time-varying communication delays between neighbors; and only a subset of the follower agents communicating with the leader. Note, however, that due to the effects of disturbances and communication delays, an attempt to satisfy the above goal may only result in the ultimately bounded result $\limsup_{t\to\infty} \|q_L(t) - q_i(t)\| \leq \varepsilon$ for some constant $\varepsilon \in \mathbb{R}_{>0}$.

III. Controller Development

Error signals used for feedback controllers in network synchronization typically take the form $e_i \triangleq \sum_{j\in\mathcal{N}_{Fi}} a_{ij}(q_j(t) - q_i(t)) + b_i(q_L(t) - q_i(t))$ (cf. [1]–[4]). However, because communication is delayed in the network, the error signal e_i is not implementable in this scenario. Alternatively, the delayed feedback error signal $e_{\tau i} \in \mathbb{R}^m$ is introduced as

$$e_{\tau i} \triangleq \sum_{j\in\mathcal{N}_{Fi}} a_{ij}\left(q_j\left(t - \tau_{ji}(t)\right) - q_i(t)\right) + b_i\left(q_L(t) - q_i(t)\right),$$

and an auxiliary delayed error signal $r_{\tau i} \in \mathbb{R}^m$ is analogously defined as

$$r_{\tau i} \triangleq \sum_{j\in\mathcal{N}_{Fi}} a_{ij}\left(\dot{q}_j\left(t - \tau_{ji}(t)\right) - \dot{q}_i(t)\right) + b_i\left(\dot{q}_L(t) - \dot{q}_i(t)\right) + \lambda e_{\tau i}, \quad (2)$$

where $\lambda \in \mathbb{R}_{>0}$ is a constant control gain. Thus, neighbors' delayed state and state derivative are to be used for control purposes with the implementable error signals $e_{\tau i}$ and $r_{\tau i}$.[1] A communication-delayed proportional-derivative (PD) controller, based on one-hop neighbor feedback, is designed for agent $i \in \mathcal{V}_F$ as

$$u_i = k r_{\tau i}, \quad (3)$$

where $k \in \mathbb{R}_{>0}$ is a constant control gain. Note that, as opposed to the controller in [20], the difference between a neighbor's delayed state and an agent's own state delayed by the same amount is not used in the controller, since the communication delay is unknown.

IV. Closed-loop error system

So that a network-wide closed-loop error system may be succinctly described, let the time-varying communication delays corresponding to each communication channel $\{\tau_{ji} \mid (j,i) \in \mathcal{E}_F\}$ be serialized by a surjective mapping to the set $\{\tau_l \mid (l \in \{1,\ldots,\Gamma\}) \wedge (\tau_l : \mathbb{R} \to \mathbb{R}_{\geq 0})\}$ such that $\tau_p(t) = \tau_q(t)\ \forall t \in \mathbb{R}$ if and only if $p = q$, where $\Gamma \in \mathbb{Z}_{>0}$ is the number of unique communication delays in the set $\{\tau_{ji} \mid (j,i) \in \mathcal{E}_F\}$.[2] In other words, each time-varying delay τ_{ji}, $(j,i) \in \mathcal{E}_F$ is equivalent to only one of the serialized delays τ_l, $l \in \{1,\ldots,\Gamma\}$, and each serialized delay τ_l, $l \in \{1,\ldots,\Gamma\}$ is equivalent to at least one of the delays τ_{ji}, $(j,i) \in \mathcal{E}_F$. This representation facilitates convenient description of communication channels which have the same delay. Additionally, let the matrix $\mathcal{A}_l \triangleq \left[\mathfrak{a}_{ij}^l\right] \otimes I_m \in \mathbb{R}^{Fm\times Fm}$ be defined with $\mathfrak{a}_{ij}^l \in \mathbb{R}$ such that $\mathfrak{a}_{ij}^l = a_{ij}$ if $\tau_{ji} \equiv \tau_l$ and $\mathfrak{a}_{ij}^l = 0$ otherwise. Note that $\sum_{l=1}^{\Gamma} \mathcal{A}_l = A \otimes I_m$; in other words, the nonzero components in $\mathcal{A}_l$ are the edge weights which correspond to communication links which all have the same communication delay τ_l. Finally, let the vector $Q_{\tau_l} \in \mathbb{R}^{Fm}$ be defined as $Q_{\tau_l}(t) \triangleq Q(t - \tau_l)$, where $Q \triangleq \left[q_1^T, \ldots, q_F^T\right]^T$.

For notational brevity, the networked systems' dynamics are conglomerated into block matrices and composite vectors as

$$\begin{aligned}
M &\triangleq \operatorname{diag}\{M_1, \ldots, M_F\} \in \mathbb{R}^{Fm\times Fm},\\
C &\triangleq \operatorname{diag}\{C_1, \ldots, C_F\} \in \mathbb{R}^{Fm\times Fm},\\
F &\triangleq \left[F_1^T, \ldots, F_F^T\right]^T \in \mathbb{R}^{Fm},\\
G &\triangleq \left[G_1^T, \ldots, G_F^T\right]^T \in \mathbb{R}^{Fm},\\
U &\triangleq \left[u_1^T, \ldots, u_F^T\right]^T \in \mathbb{R}^{Fm},\\
d &\triangleq \left[d_1^T, \ldots, d_F^T\right]^T \in \mathbb{R}^{Fm},
\end{aligned}$$

such that

$$M(Q)\ddot{Q} + C\left(Q,\dot{Q}\right)\dot{Q} + F\left(\dot{Q}\right) + G(Q) = U + d(t). \quad (4)$$

Non-implemented error signals $E \triangleq Q_L - Q \in \mathbb{R}^{Fm}$ and $R \triangleq \dot{E} + \lambda E \in \mathbb{R}^{Fm}$ are introduced to develop a network-wide closed-loop error system, where $Q_L \triangleq \underline{1}_F \otimes q_L \in \mathbb{R}^{Fm}$ and $\underline{1}_F$ is an F-dimensional column vector of ones. Clearly, if $\|E\| \to 0$, then the control objective is achieved.

By taking the time-derivative of R and premultiplying by the block inertia matrix M, the closed-loop error system is represented using (3) and (4) as

$$M\dot{R} = C\dot{Q} + F + G - d + M\ddot{Q}_L + \lambda M\dot{E} - kR_\tau, \quad (5)$$

where

$$\begin{aligned}
R_\tau &\triangleq \left[r_{\tau 1}^T, \ldots, r_{\tau m}^T\right]^T\\
&= \sum_{l=1}^{\Gamma} \mathcal{A}_l\left(\dot{Q}_{\tau_l} + \lambda Q_{\tau_l}\right) - \left((D+B)\otimes I_m\right)\left(\dot{Q} + \lambda Q\right)\\
&\quad + (B\otimes I_m)\left(\dot{Q}_L + \lambda Q_L\right).
\end{aligned}$$

Throughout the rest of the paper, functional dependency is omitted where the meaning is clear. After using the fact that

[1] It is assumed that a neighbor's state derivative is measured by that neighbor and then communicated; i.e., this approach does not solve the communication-delayed output feedback problem. Numerical computation of the delayed state derivative would be skewed by effects of the time-varying delay.

[2] This approach does not omit the case in which some communication links have no delay.

$\left(\dot{Q}_L + \lambda Q_L\right) \in \mathrm{Null}\left(\mathcal{L}_F \otimes I_m\right)$ due to the structure of the Laplacian matrix and using the First Fundamental Theorem of Calculus, (5) may be re-expressed as

$$M\dot{R} = C\dot{Q} + F + G - d + M\ddot{Q}_L + \lambda M\dot{E} - kL_B R + k\sum_{l=1}^{\Gamma} \mathcal{A}_l \Upsilon_l, \tag{6}$$

where $\Upsilon_l \triangleq \int_{t-\tau_l}^{t} \left(\ddot{Q}(s) + \lambda \dot{Q}(s)\right) ds$. The terms $C\dot{Q}, F, G, d, M\ddot{Q}_L, \lambda M\dot{E}$ in (6) can be compensated for using robust control methods; however, compensating for the term $k\sum_{l=1}^{\Gamma}\mathcal{A}_l\Upsilon_l$ is difficult due to the delayed state and multiplication by the gain k. The following sections demonstrate that the decentralized controller in (3) yields convergence to a neighborhood around the leader state for each follower agent despite this delay-contributing term for small enough time-varying heterogeneous network delays.

V. Stability analysis

Before the stability analysis is presented, some facilitating constants, functions, and sets are introduced. Let $c \in \mathbb{R}_{>0}$ denote a tunable constant parameter and let the constant $\underline{c} \in \mathbb{R}$ be defined as $\underline{c} \triangleq c\left(\lambda - \frac{1}{2}\right)$. Also let the constants $\phi_1, \phi_2 \in \mathbb{R}_{\geq 0}$ be tunable parameters, where

$$\phi_1 \geq 1. \tag{7}$$

Let the auxiliary constants $\underline{k}, \eta, \theta \in \mathbb{R}$ be defined as $\underline{k} \triangleq k\left(\lambda_{\min}(L_B) - \frac{\bar{\tau}k^3}{2}\right) - \frac{c}{2}$, $\eta \triangleq \min\left\{\frac{\underline{c}}{2}, \frac{\underline{k}}{6}\right\}$, and $\theta \triangleq \min\left\{\eta, \frac{\phi_1}{2\bar{\tau}}, \frac{\phi_2}{2}, \frac{\phi_2}{2\bar{\tau}}\right\}$, where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue. The stability analysis is constructed with the state $y \in \mathbb{R}^{2Fm+3}$ defined as the composite vector[3] $y \triangleq \left[Z^T, \Psi_1^{\frac{1}{2}}, \Psi_2^{\frac{1}{2}}, \Psi_3^{\frac{1}{2}}\right]^T$, where $Z \in \mathbb{R}^{2Fm}$ is the composite error vector $Z \triangleq \begin{bmatrix} E^T & R^T \end{bmatrix}^T$, and Ψ_1, Ψ_2, Ψ_3 denote LK functionals defined as

$$\Psi_1 \triangleq \frac{\Gamma\bar{A}}{k^2}\int_{t-\bar{\tau}}^{t}\int_{s}^{t}\left\|\ddot{Q}(\sigma) + \lambda\dot{Q}(\sigma)\right\|^2 d\sigma ds,$$

$$\Psi_2 \triangleq \frac{\bar{\tau}\Gamma^2\bar{A}}{\underline{m}\omega}\left(\frac{2(k+1)}{k\underline{m}} + \frac{1}{k}\right) \cdot \sum_{l=1}^{\Gamma}\int_{t-\tau_l}^{t}\left\|\mathcal{A}_l\left(\dot{Q}(\sigma) + \lambda Q(\sigma)\right)\right\|^2 d\sigma,$$

$$\Psi_3 \triangleq \frac{\bar{\tau}\Gamma^2\bar{A}}{\underline{m}\omega}\left(\frac{2(k+1)}{k\underline{m}} + \frac{1}{k}\right) \cdot \sum_{l=1}^{\Gamma}\int_{t-\tau_l}^{t}\int_{s}^{t}\left\|\mathcal{A}_l\left(\dot{Q}(\sigma) + \lambda Q(\sigma)\right)\right\|^2 d\sigma ds,$$

where $a \cdot b$ denotes standard multiplication for $a, b \in \mathbb{R}$, $\bar{A} \triangleq \lambda_{\max}\left(\sum_{l=1}^{\Gamma}\mathcal{A}_l\mathcal{A}_l^T\right)$, $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue, and the unknown constant $\omega \in \mathbb{R}$ is defined as $\omega \triangleq 1 - \sup_{t\in\mathbb{R}, l\in 1,\ldots,\Gamma}\dot{\tau}_l(t)$, which is positive by Assumption 5.

To facilitate the description of the UUB result in the following stability analysis, the constants $N_{d0}, N_{d1}, N_{d2} \in \mathbb{R}_{\geq 0}$ are defined as

$$N_{d0} \triangleq \bar{d} + \bar{m}\sup_{t\in\mathbb{R}}\left\|\ddot{Q}_L\right\| + \sup_{t\in\mathbb{R}}\left\|S_0\left(Q_L, \dot{Q}_L\right)\right\|, \tag{8}$$

$$N_{d1} \triangleq \frac{2\Gamma\bar{A}}{k\underline{m}^2}\lambda_{\max}(L_B)\,\bar{d} + \sup_{t\in\mathbb{R}}\left\|S_1\left(Q_L, \dot{Q}_L\right)\right\|, \tag{9}$$

$$N_{d2} \triangleq \frac{\Gamma\bar{A}\bar{d}^2}{\underline{m}^2}\left(\frac{1}{k^2} + \frac{1}{k}\right) + \sup_{t\in\mathbb{R}}\left|S_2\left(Q_L, \dot{Q}_L\right)\right|, \tag{10}$$

and the functions $\tilde{N}_0 : \Pi_{p=1}^{6}\mathbb{R}^{Fm} \to \mathbb{R}^{Fm}$, $\tilde{N}_1 : \Pi_{p=1}^{5}\mathbb{R}^{Fm} \to \mathbb{R}^{Fm}$, $\tilde{N}_2 : \Pi_{p=1}^{4} \to \mathbb{R}$ are defined as

$$\tilde{N}_0 \triangleq S_0\left(Q, \dot{Q}\right) - S_0\left(Q_L, \dot{Q}_L\right) + f_0\left(Q, \dot{Q}, E, R\right),$$

$$\tilde{N}_1 \triangleq S_1\left(Q, \dot{Q}\right) - S_1\left(Q_L, \dot{Q}_L\right) + \frac{2\Gamma\bar{A}}{\underline{m}^2}L_B L_B R,$$

$$\tilde{N}_2 \triangleq S_2\left(Q, \dot{Q}\right) - S_2\left(Q_L, \dot{Q}_L\right),$$

where the auxiliary functions $S_0 : \mathbb{R}^{Fm}\times\mathbb{R}^{Fm} \to \mathbb{R}^{Fm}$, $S_1 : \mathbb{R}^{Fm}\times\mathbb{R}^{Fm} \to \mathbb{R}^{Fm}$, $S_2 : \mathbb{R}^{Fm}\times\mathbb{R}^{Fm} \to \mathbb{R}_{\geq 0}$, $f_0 : \Pi_{p=1}^{4}\mathbb{R}^{Fm} \to \mathbb{R}^{Fm}$ are defined as

$$S_0\left(Q, \dot{Q}\right) \triangleq C\dot{Q} + F + G,$$

$$S_1\left(Q, \dot{Q}\right) \triangleq -\frac{2\Gamma\bar{A}}{k}L_B M^{-2}S_0\left(Q, \dot{Q}\right) + \frac{2\Gamma\bar{A}}{k}\lambda L_B M^{-1}\dot{Q} + 2\Gamma\bar{A}L_B M^{-2}\sum_{l=1}^{\Gamma}\mathcal{A}_l\left(\dot{Q} + \lambda Q\right),$$

$$\begin{aligned} S_2\left(Q, \dot{Q}\right) \triangleq \Gamma\bar{A}\Bigg[&\left(\frac{1}{k^2\underline{m}^2} + \frac{1}{k\underline{m}^2}\right)\left\|S_0\left(Q, \dot{Q}\right)\right\|^2 + \left(\frac{2}{k^2\underline{m}^2}\bar{d} + \frac{2\lambda}{k^2\underline{m}}\left\|\dot{Q}\right\|\right)\left\|S_0\left(Q, \dot{Q}\right)\right\| \\ &+ \frac{2}{k^2\underline{m}}\lambda\left\|\dot{Q}\right\|\bar{d} + \left(\frac{2}{k\underline{m}^2}\left\|S_0\left(Q, \dot{Q}\right)\right\| + \frac{2\lambda}{k\underline{m}}\left\|\dot{Q}\right\|\right) \cdot \left\|\sum_{l=1}^{\Gamma}\mathcal{A}_l\left(\dot{Q} + \lambda Q\right)\right\| + \left(\frac{\lambda^2}{k^2} + \frac{1}{k\underline{m}}\lambda^2\right)\left\|\dot{Q}\right\|^2 \\ &+ \left(\frac{1}{\underline{m}^2} + \frac{2}{k\underline{m}^2}\bar{d}\right)\left\|\sum_{l=1}^{\Gamma}\mathcal{A}_l\left(\dot{Q} + \lambda Q\right)\right\|^2\Bigg] \\ &+ \frac{\Gamma^2\bar{A}}{\underline{m}\omega}\left(\frac{2(k+1)}{k\underline{m}} + \frac{1}{k}\right)\sum_{l=1}^{\Gamma}\left\|\mathcal{A}_l\left(\dot{Q} + \lambda Q\right)\right\|^2, \end{aligned}$$

$$f_0\left(Q, \dot{Q}, E, R\right) \triangleq \lambda MR - \lambda^2 ME + \frac{1}{2}\dot{M}R.$$

[3]The LK functionals are interpreted as time-varying signals and are incorporated into the overall system state to facilitate the stability analysis.

The signals $\tilde{N}_0, \tilde{N}_1, \tilde{N}_2$ contain terms which can be upper-bounded by a function of the error signals E and R. By [25, Lemma 5], there exist strictly increasing, radially unbounded functions $\rho_0, \rho_1, \rho_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which upper-bound $\tilde{N}_0, \tilde{N}_1, \tilde{N}_2$ as[4]

$$\left\|\tilde{N}_0\right\| \leq \rho_0\left(\|Z\|\right)\|Z\|, \quad \left\|\tilde{N}_1\right\| \leq \rho_1\left(\|Z\|\right)\|Z\|, \quad \left|\tilde{N}_2\right| \leq \rho_2\left(\|Z\|\right)\|Z\|, \tag{11}$$

where the bound for $\left\|\tilde{N}_0\right\|$ is facilitated by adding and subtracting the expression $f_0\left(Q_L, \dot{Q}_L, E, R\right)$ in $\tilde{N}_0$.

The set $\mathcal{D} \subset \mathbb{R}^{2Fm+3}$ is defined as

$$\mathcal{D} \triangleq \left\{\xi \in \mathbb{R}^{2Fm+3} \mid \|\xi\| < \inf\left\{\rho^{-1}\left(\left[\sqrt{\eta}, \infty\right)\right)\right\}\right\},$$

where $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a strictly increasing, radially unbounded function defined as

$$\rho\left(\|Z\|\right) \triangleq \left(\frac{3\rho_0^2\left(\|Z\|\right)}{2\underline{k}} + \frac{3\bar{\tau}^2\phi_1^2\rho_1^2\left(\|Z\|\right)}{2\underline{k}} + \bar{\tau}\left(\phi_1 + \bar{\tau}\phi_2\right)\rho_2^2\left(\|Z\|\right)\right)^{\frac{1}{2}}, \tag{12}$$

and the inverse image $\rho^{-1}(\Theta) \subset \mathbb{R}$ for a set $\Theta \subset \mathbb{R}$ is defined as $\rho^{-1}(\Theta) \triangleq \{\xi \in \mathbb{R} \mid \rho(\xi) \in \Theta\}$. The stabilizing set of initial conditions, $\mathcal{S}$, is defined as

$$\mathcal{S} \triangleq \left\{\xi \in \mathcal{D} \mid \|\xi\| < \sqrt{\frac{\min\left\{\frac{c}{2}, \frac{\underline{m}}{2}, \phi_1, \phi_2\right\}}{\max\left\{\frac{c}{2}, \frac{\bar{m}}{2}, \phi_1, \phi_2\right\}}} \cdot \inf\left\{\rho^{-1}\left(\left[\sqrt{\eta}, \infty\right)\right)\right\}\right\}.$$

The following assumption provides a sufficient condition for the subsequent stability analysis by describing how small the network communication delays should be to ensure stability for a given network configuration.

Assumption 6. For a given network graph $\mathcal{G}$, the communication delay upper bound $\bar{\tau} > 0$ is sufficiently small such that there exists a selection for the gain k such that $\bar{\tau} < \frac{2\lambda_{\min}(L_B)}{k^3}$, $\mathcal{S} \neq \emptyset$, and

$$\sqrt{\frac{\min\left\{\frac{c}{2}, \frac{\underline{m}}{2}, \phi_1, \phi_2\right\}}{\max\left\{\frac{c}{2}, \frac{\bar{m}}{2}, \phi_1, \phi_2\right\}}} \inf\left\{\rho^{-1}\left(\left[\sqrt{\eta}, \infty\right)\right)\right\} > \sqrt{\frac{2}{\theta}}\left(\frac{3\left(N_{d0}^2 + \bar{\tau}^2\phi_1^2 N_{d1}^2\right)}{2\underline{k}} + \bar{\tau}\left(\phi_1 + \bar{\tau}\phi_2\right)\left(N_{d2} + \frac{1}{4}\right)\right)^{\frac{1}{2}}. \tag{13}$$

Remark 1. Assumption 6 ensures that there exists a selection for c such that $\underline{k} > 0$. Accordingly, let the value for c be assigned such that $0 < c < k\left(2\lambda_{\min}(L_B) - \bar{\tau}k^3\right)$. Assumption 6 also ensures that there exist stabilizing initial conditions and that the uniform ultimate bound on the convergence of each agent toward the leader state is within the considered domain $\mathcal{D}$.

Remark 2. Due to the presence of $\underline{k}$ and $\bar{\tau}$ in (12), there exists a sufficiently small value for $\bar{\tau}$ such that there exists a sufficiently large gain k such that $\bar{\tau} < \frac{2\lambda_{\min}(L_B)}{k^3}$ and $\inf\left\{\rho^{-1}\left(\left[\sqrt{\eta}, \infty\right)\right)\right\} > \delta$ for any $\delta \in \mathbb{R}_{>0}$, i.e., the set $\mathcal{S}$ can contain any initial condition for a small enough delay upper bound $\bar{\tau}$.

Theorem 1. *The communication-delayed controller in (3) provides UUB synchronization for a network of agents with dynamics given by (1) in the sense that* $\limsup_{t\to\infty}\|q_L(t) - q_i(t)\| \leq \varepsilon$ *for every follower agent* $i \in \mathcal{V}_F$ *for all* $y(0) \in \mathcal{S}$, *provided that Assumptions 1-6 are satisfied and the gain* λ *satisfies*

$$\lambda > \frac{1}{2}, \tag{14}$$

where the constant $\varepsilon \in \mathbb{R}$ *is defined as*

$$\varepsilon \triangleq \sqrt{\frac{2\max\left\{\frac{c}{2}, \frac{\bar{m}}{2}, \phi_1, \phi_2\right\}}{\theta\min\left\{\frac{c}{2}, \frac{\underline{m}}{2}, \phi_1, \phi_2\right\}}}\left(\frac{3\left(N_{d0}^2 + \bar{\tau}^2\phi_1^2 N_{d1}^2\right)}{2\underline{k}} + \bar{\tau}\left(\phi_1 + \bar{\tau}\phi_2\right)\left(N_{d2} + \frac{1}{4}\right)\right)^{\frac{1}{2}}.$$

Proof: (Sketch) Consider the candidate Lyapunov function $V_L : \mathcal{D} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined as

$$V_L \triangleq \frac{c}{2}E^T E + \frac{1}{2}R^T M R + \phi_1\Psi_1 + \phi_1\Psi_2 + \phi_2\Psi_3, \tag{15}$$

which satisfies the inequalities

$$V_L(y,t) \geq \min\left\{\frac{c}{2}, \frac{\underline{m}}{2}, \phi_1, \phi_2\right\}\|y\|^2,$$
$$V_L(y,t) \leq \max\left\{\frac{c}{2}, \frac{\bar{m}}{2}, \phi_1, \phi_2\right\}\|y\|^2,$$

for all $y \in \mathbb{R}^{2Fm+3}$ and $t \in \mathbb{R}$, where the block inertia matrix M is interpreted as a function of time. By using the closed-loop error system in (6), the Leibniz rule, Young's inequality, nonlinear damping, and the inequality

$$kR^T\sum_{l=1}^{\Gamma}\mathcal{A}_l\Upsilon_l \leq \frac{\bar{\tau}k^4}{2}R^T R + \frac{\Gamma\bar{A}}{2k^2}\int_{t-\bar{\tau}}^{t}\left\|\ddot{Q}(s) + \lambda\dot{Q}(s)\right\|^2 ds,$$

the derivative of the Lyapunov function can be upper-bounded as

$$\dot{V}_L \leq -\frac{\theta}{2}\|y\|^2 \quad \forall\|y\| \geq \sqrt{\frac{2}{\theta}}\left(\frac{3\left(N_{d0}^2 + \bar{\tau}^2\phi_1^2 N_{d1}^2\right)}{2\underline{k}} + \bar{\tau}\left(\phi_1 + \bar{\tau}\phi_2\right)\left(N_{d2} + \frac{1}{4}\right)\right)^{\frac{1}{2}} \tag{16}$$

for all $y \in \mathcal{D}$. By (16), [26, Theorem 4.18] and Assumption 6,

$$\limsup_{t\to\infty}\|q_i(t) - q_L(t)\| \leq \limsup_{t\to\infty}\|y(t)\| \leq \varepsilon$$

[4]While the *smallest* upper-bounding functions of the dynamics in $\tilde{N}_0, \tilde{N}_1, \tilde{N}_2$ may not be known, the bounding functions ρ_0, ρ_1, ρ_2 may feasibly be constructed; for example, a friction coefficient may be unknown, but a sufficient upper bound can easily be determined.

for all $i \in \mathcal{V}_F$ and $y(0) \in \mathcal{S}$, since $\|q_L - q_i\| \leq \|Q_L - Q_i\| = \|E\| \leq \|y\|$ for all $i \in \mathcal{V}_F$. Hence, since $y, q_L, \dot{q}_L \in \mathcal{L}_\infty$, it is clear that $q_i, \dot{q}_i \in \mathcal{L}_\infty$ for all $i \in \mathcal{V}_F$, and the control effort is bounded during the entire state trajectory. ■

Although this stability analysis only provides sufficient conditions, the restriction in Assumption 6 and the UUB nature of the result in Theorem 1 correspond with several intuitive notions about communication-delayed networked systems:

- communication-delayed systems may not be stable for arbitrarily large gains in proportional and derivative feedback control,
- a larger communication delay may shrink the set of stabilizing initial conditions and increase the ultimate upper bound of the norm of the tracking error trajectory,
- quickly varying communication delays may shrink the set of stabilizing initial conditions and increase the ultimate upper bound of the tracking error trajectory (due to the presence of ω in S_2).

Furthermore, as the delay upper bound tends toward zero (ignoring the singularity of $\bar{\tau} \equiv 0$, which obviates the need for the LK functional-based approach taken in this paper), the effects of the delay vanish and the stability analysis resembles that of a high-gain robust control analysis, such as in [4].

Additionally, note that while the controller in (3) is decentralized in communication, gain selection is a centralized process that occurs prior to controller implementation.

VI. Conclusion

A stability analysis is presented which provides sufficient conditions for UUB leader-synchronization of a network of communication-delayed agents using a decentralized neighborhood-based PD controller. The agents are modeled with dynamics described by heterogeneous, uncertain Euler-Lagrange equations of motion affected by time-varying, unknown exogenous disturbances. The communication delay is considered to be heterogeneous, time-varying, and uncertain. Salient dependencies for the sufficient conditions for stability in synchronization are the upper bound of the heterogeneous communication delays, feedback gains, and network connectivity. Some prominent assumptions are that there is no delay in communication from the network leader, the dynamics and input disturbances are sufficiently smooth, the follower communication network is undirected, and at least one follower agent receives information from the leader.

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